Probability and Statistics: Midterm 1 Study Guide

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1 Introduction to Probability

1. **Probability Measure:** A probability measure on (Ω, F) is a function P from f to the interval [0,1] satisfying the following conditions:
   (a) $P(\Omega) = 1$
   (b) If $A_1, A_2, ...$ are pairwise disjoint in F, so that $A_j \cap A_j = \emptyset$ whenever $i \neq j$, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(\bigcup_{i=1}^{\infty} A_i)$
2. **Probability Space:** The triple (Ω, F, P) of objects where:
   (a) Ω is a set,
   (b) F is an event space made of subsets of Ω,
   (c) P is a **Probability Measure** on (Ω, F)

1. **Multiplication Principle**: If an experiment has m outcomes and a second experiment has n outcomes, there are mn possible outcomes for the two experiments.
2. **Extended Multiplication Principle:** If there are p experiments with $n_1 n_2 ... n_p$ possible outcomes, the total of the possible outcomes is $n_1, n_2 ... n_p$ possible outcomes for the p experiments.

2 Conditional Probability

1. If we have a Big Omega $= \{1, 2, 3, 4, 5, 6\}$ in which we want to find out what is the outcome we will get a specific number (with one roll), it will always be equal to $\frac{1}{6}$
2. **Calculation for Conditional Probability:** $P(A|B) = \frac{P(A \cap B)}{P(B)}$
3. **Calc. Conditional Prob. Extended:** $P(A|B) = \frac{P(A|B)P(B)}{P(B)}$
4. **Bayes Rule:** $P(B_i|A) = \frac{P(A \cup B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{\infty} P(A|B_j)}$

   Ex: Suppose you were given the following information:

In the absence of any special information, the probability that a patient of a certain age has breast cancer is 1%. *If the patient has breast cancer, the probability the radiologist will correctly diagnose it is* 80%, $(P(+|B) = 80\%)$ *and* $(P(-|B) = 20\%)$. *If the patient has a benign lesion, the probability that the radiologist will incorrectly diagnose it is* 10%, *so* $P(-|B^c) = 10\%$, *and* $P(+|B^c) = 90\%$.

Q: That is the probability that a patient with a positive mammogram actually has breast cancer? **Find:** $P(B|+)$

Using Bayes Rule, we obtain the equation: $P(B|+) = \frac{P(+|B)P(B)}{P(+|B)P(B)+P(+|B^c)P(B^c)}$

3 Law of Total Probability

1. **The Law of Total Probability**(Partition Thm.): $\{B_1, B_2, ...\}$ is a **partition** of Ω if $\cup_i B_i = \Omega$ and $B_i \cap B_j = \emptyset$ wherever $i \neq j$.

4 Independence

1. **Independence:** Events A and B of a probability space (Ω, F, P) are independent if $P(A \cap B) = P(A)P(B)$ – they are dependent otherwise.

2. **Mutual Independence:** Occurs if the events $A_1, A_2, ...$, $P(\cap_j A_{i_j}) = \pi_j P(A_{i_j})$ for any sub collection $A_{i1}, A_{i2}, ...$

3. **Pairwise Independence:** occurs if $P(A_i \cap A_j) = P(A_i)P(A_j)$ whenever $i \neq j$.

   Events A, B, C are **pairwise independent**:

   $P(A)P(B) = P(A \cap B), P(A \cap C) = P(A)P(C)$.

Note: They are **Mutually Independent** *iff* $P(A \cap B \cap C) = P(A)P(B)P(C)$

5 Discrete Random Variables:

$$X = \begin{cases} -1 & P(-1) = \frac{1}{2} \\ 1 & P(1) = \frac{1}{2} \end{cases}$$

We have a random variable with values -1 and 1, each with probability $\frac{1}{2}$.

To make a d.r.v to determine the different faces we can roll on a dice, it would be as follows..

$$X = \begin{cases} 1 & P(1) = \frac{1}{6} \\ 2 & P(2) = \frac{1}{6} \\ 3 & P(3) = \frac{1}{6} \\ 4 & P(4) = \frac{1}{6} \\ 5 & P(5) = \frac{1}{6} \\ 6 & P(6) = \frac{1}{6} \\ 0 & otherwise \end{cases}$$

This outlines the possibilities of each number with each roll

Note: The term **discrete** refers to the condition that $X(\Omega)$ is countable.

Probability Mass Function: The *pmf* $p_X(x)$ is the function

$$p_X : R \to [0, 1]$$

defined by the function $p_X(x) = P(\{\omega \in \Omega : X(\omega) = x\})$

ex. Flip a fair coin and define...

$$Y = \begin{cases} -1 & iff = T \\ 1 & iff = H \end{cases}$$

(a) The probability mass function of p_Y of Y is given by $P_Y(1) = P_Y(-1) = \frac{1}{2}$. The probability of obtaining either is equal to $\frac{1}{2}$.

Cumulative distribution function: of a random variable X on the probability space (Ω, F, P) is the function $F_X : R \to [0, 1]$, defined by

$$F(X) = F_X(x) = P(X \leq x) = P(\{\omega \in \Omega : X(\omega) \leq x\})$$

NOTE: The **cumulative distribution function** is characterized by...

(1.) $F_X(x) \leq F_X(y)$ if $x \leq y$,

(2.) $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$

(3.) $F_X(x)$ is continuous from the right, that is:

$$\lim_{\epsilon \to 0^+} F_X(x + \epsilon) = F_X(x)$$

(1.) $\to$ Property (1.) holds becuase

$$\{\in \Omega : X(\omega) \leq x\} \subseteq \{\in \Omega : X(\omega) \leq y\}$$

Wherever, $x \leq y$.

(2.) $\to$ Property (2.) must be proved.

Ex: Proving Cumulative Distribution Function...

Given: $\frac{1}{\pi}tan^{-1}(x) + \frac{1}{2}$, prove that it is a cumulative distribution function by checking the three conditions that characterize it.

(1.) $F'(x) = \frac{1}{\pi}\frac{1}{1+x^2} \rightarrow$ **Cauchy Density Function**

(2.) $\lim_{x\to-\infty} \frac{1}{\pi}tan^{-1}(x) + \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} = 0$

$\lim_{x\to\infty} \frac{1}{\pi}tan^{-1}(x) + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$

(3.) F is continuous because tan^{-1} is a continuous function.

5.1 Functions of Discrete Random Variables:

If X is a discrete random variable on (Ω, F, P), and $g : R \rightarrow R$ is a function, then $Y = g(X)$ defined by $Y(\omega) = g(X(\omega)), \forall\omega \in \Omega$ is duly a discrete random variable on (Ω, F, P).

Note: The **pmf** of Y is given by

$$p_Y(y) = P(Y = y) = P(X \in g^{-1}(y)) = \sum_{x^{-1}\cap X(\Omega)} P(X = x)$$

ex1:

$$g(x) = ax + b$$

$$Y = g(X) = ax + b$$

$$p_Y(Y) = P(X = \frac{y-b}{a}, \forall y \in R$$

ex2:

$$g(x) = cx^2$$

$$Y = g(X) = cX^2$$

$$p_Y(Y) = p_Y(y) = \begin{cases} P(X = \sqrt{\frac{y}{c}}) + P(-\sqrt{\frac{y}{c}}) & y > 0 \\ P(X = 0) & y = 0 \\ 0 & y < 0 \end{cases}$$

6 Expectation and Variance:

Def: **Mean** or **Expectation** of X is defined to be:

$$E(X) = \sum_{x\in X(\Omega)} xP(X = x)$$

$$= \sum_{x\in X(\Omega)} xp_X(x)$$

if the the sum converges absolutely

Thm: If X is a discrete random variable and $g : R \to R_1$, then the ugly summation of g(x) is worked into the problem as follows...

$$\sum_{x \in X(\Omega)} g(x)P(X = x)$$

also known as:

$$\sum_{x \in X(\Omega)} g(x)p_X(x)$$

Def: **Variance** of X is defined as follows:

$$Var(X) = E(X^2) - [E(X)]^2$$

it can be further *complicated* into

$$Var(X) = E[(X - E(X)^2]$$

or

$$E[X^2 - 2XE(X) + E(X)^2],$$

however for the most part it'd be best to just use the initial definition.

6.1 Discrete Random Variables - Expectation and Variance (oh how fun):

1. **Bernoulli Distribution:** if $X(\Omega) = (0, 1)$ annnnnd, $P(X = 0) = 1 - p$, annnddd $P(X = 1) = p$, then the Expectation is p. The Variance, using the formula in Section 6 is

$$Var(X) = E(X^2) - E(X)^2 = p - p^2 = p(1 - p)$$

.

2. **Binomial Distribution:** $p_X(x) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$.

Note:

$$\sum_{k=0}^{n} P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} = [1 + (1 - p)]^n = 1$$

Expectation: The expectation of any binomial random variable is **ALWAYS** $E(X) = np$.

Variance: The variance of any binomial random variable is **ALWAYS**

$$Var(X) = npq, q = (1 - p).$$

6.2 Discrete Random Variables - Continuous Random Variables (oh how much fun):

A **random variable** X on the proabbility space (Ω, F, P) is a functino $X : \Omega \to R$ such that

$$\{\omega \in \Omega : X(\omega) \leq x\} \in F =, \forall x \in R$$

6.3 Density Function:

Typically, continuous random variables that we're interested in have a **density function** of f, such that $f(x)$ is greater than or equal to 0 for all real numbers.

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du, \forall x \in R$$

If X has a **continuous** density function f, the following properties are evident:

$$P(a \leq X \leq b) = F_x(b) - F_X(a) = \int_a^b f(u)du$$

$$\int_{-\infty}^{\infty} f(u)du = 1$$

$$f(x) = \frac{d}{dx} F_X(x)$$

ex: The simplest density function is the **uniform density function**:

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & x < a \cup x > b \end{cases}$$

ex: As follows is the uniform density on on interval [a,b].

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

Note: $F_X(x) = \int_{-\infty}^{x}$ and $\int_{-\infty}^{\infty} f(u)du = 1$. X is a uniform random variable on interval $[0, 1]$.

6.4 Joint Distributions:

1. If X and Y are d.r.v's, the **joint(probability) mass function** of X and Y is the function $p_{X,Y} : R^2 \rightarrow [0, 1]$ defined by the function:

$$p_{X,Y}(x, y) = P(\{\omega \in \Omega : X(\omega) = x, Y(\omega) = y\})$$

Marginal Mass Function: The marginal mass function refers to "pulling" $p_X(x)$ and $p_Y(y)$ out of the joint function.