Lecture 9 - Discrete Mafematiks

Kori Vernon

July 28, 2020

1 Proofs By Smallest Counterexample

Template

1. Let x be a smallest counterexample to the result to which we need to prove.
   We need to show or indicate that such x exists there
2. Rule out x being in the very smallest possibility (**Basis Step**)
3. Consider an instance $x*$ (usually $x* = x - 1$) of the result that is "just" smaller than x.
   Where $x* = x - 1$ is NOT a counterexample, so statement for $x*$ is true, we show that it leads us to a contradiction about x. $\Rightarrow\Leftarrow$
   Thus, conclude that the statement is true.

1.1 Examples:

1. Let n be a positive integer. The sum of the first n odd numbers is n^2.

Proof: Consider some $n \in \mathbb{N}$, then $1 + 3 + 5 + ... + (2n - 1) = n^2$

- Suppose this claim is false, then $\exists x \in \mathbb{N} : 1 + 3 + 5 + ... + (2x - 1) \neq x^2$, where x is the smallest natural number.
- Note that $x \neq 1$, since $1 = 1^2$, so $x > 1$. Then for $x - 1$, the statement is true.
- Thus, $1 + 3 + 5 + ... + (2x - 1) = (x - 1)^2$.
  We add $2x - 1$ to both sides, so:

$$1 + 3 + 5 + ... + (2x - 1) + (2x - 1) = (x - 1)^2 + (2x - 1)$$

$$1 + 3 + 5 + ... + (2x - 1) + (2x - 1) = x^2 - 2x + 1 + 2x - 1 \ \Rightarrow\Leftarrow$$

2 Well-Ordering Principle

Every nonempty set of natural numbers contains a least element.

Consider $P = \{x \in \mathbb{N} : n \text{ is prime}\}$, so the smallest element of P is $x = 2$.

Proof by the Well-Ordering Principle:

To prove a statement about natural numbers:

- For the sake of contradiction, assume that the statement is false. Let $X \subseteq \mathbb{N}$ be the set of counterexamples to the statement, so $X \neq \emptyset$
- Since $X \subseteq \mathbb{N}$, then X contains the least element $x \in X$.
  1. Show that $x \neq 0$
  2. Consider $x - 1$. Since $x > 0$, then $x - 1 \in \mathbb{N}$, and the statement is true for $x - 1$, because $x - 1 < x$
  3. Carry out the argument and show that the statement is true for x as well, so we reached a contradiction. $\Rightarrow\Leftarrow$.

Rk: We need to define $X =$the set of all counterexamples and $x \in X$ is the smallest counterexample.

2.1 Examples:

1. Let $n \in \mathbb{N}$. If $a \neq 0$ and $a \neq 1$, then $a^0 + a^1 + a^2 + ... + a^n = \frac{a^{n+1}-1}{a-1}$

Proof: Let X be the set of all counterexamples, so

$$X = \{n \in \mathbb{N} : \sum_{k=0}^{n} a^k \neq a^{n+1} - 1a - 1\}$$

- For the sake of contradiction, assume that the statement is false. So $X \neq \emptyset$ and $x \subseteq \mathbb{N}$, so $\exists x \in X$: by the **WOP**, x is the smallest element in X.
- Then for $n = 0$, $1 = \frac{a^{0+1}-1}{a-1} = 1$. THIS IS TRUE, SO $x \neq 0$, so $x > 0$.
- Now, consider $x - 1 \in \mathbb{N}$, and $x - 1 \not\in X$, so for "$x - 1$" the statement is true.
- Now add $a^0 + a^1 + ... + a^{(x-1)} = \frac{a^(x-1)+1}{a-1} = \frac{a^x-1}{a-1}$.
- Now, add a^x to both sides.

$$a^0 + a^1 + a^2 + ... + a^{x-1} + a^x = \frac{a^x - 1}{a-1} + a^x = \frac{a^x - 1 + a^x \cdot (a-1)}{a-1} = \frac{-1 + a^{x+1}}{a-1}$$

- x satisfies the proposition, so x is NOT a counterexample.

2. $\forall n \geq 5$, $2^n > n^2$, for $n \exists \mathbb{Z}$.

- For the sake of contradiction, assume the statement is false.

Proof: Let X be the set of all counterexamples, so $X = \{n \in \mathbb{Z} : n \geq 5, 2^n \not> n^2\}$, so $X = \emptyset$.

- By the **WOP**, X contains the least element.
- Let $x \in X$, so that x is the least element. $x \neq 5$, because $2^5 = 32 > 5^2 = 25$, so 5 is not a counterexample, so $x > 5$, so $x \geq 6$. Now consider $x - 1$, so $x - 1 \geq 5$ for which proposition is true, so

$$2^{x-1} > (x-1)^2$$

$$2^x \cdot \frac{1}{2} > (x^2 - 2x + 1) \cdot 2$$

$$2^x > 2^{x2} - 4x + 2 > 2x^2 > x^2$$

- Thus, $2x^2 - 4x + 2 \geq x^2$

$$x^2 - 4x + 2 > 0$$

$$2^2 - 4x + 4 > 2$$

$$(x-1)^2 > 2$$

$$x - 2 > 2$$

$$x > 4$$

Thus, $2^x > 2x^2 - 4x + 2 > x^2$

Fibonacci Sequence is a recursive sequence, so that the terms of a sequence depend on the previous two terms.

3. For all $n \in \mathbb{N}$, for Fibonacci sequence, $a_n \leq 1.7^n$

Proof: For the sake of contradiction, suppose this proposition is false.

Let X be the set of all counterexamples, so $X = \{n \in \mathbb{N} : a_n \not{l}eq 1.7^n\}$, so $X \neq \emptyset$

- Then, by **WOP**, Xcontains the least element.
- If $x \neq 0$, then $a_0 = 1 = 1.7^0$ ✓
- If $x \neq 1$, then $x \neq 1 = 1.7^1 \geq 1$ ✓
- Therefore, $x \geq 2$. Then for $x - 1$ and $x - 2$ proposition is true, so $a_{x-1} \leq 1.7^{x-1}$ and $a_{x-1} \leq 1.7^{x-2}$.

- Since $a_x = a_{x-2} + a_{x-1} \leq 1.7^{x-2} + 1.7^{x-1}$

$$\leq 1.7^{x-2}(1.7+1)$$

$$\leq 1.7^{x-2}(2.7)$$

$$\leq 1.7^{x-2}(2.89)$$

$$\leq 1.7^{x-2}(1.7)^2 = 1.7^x \Rightarrow\Leftarrow$$

- Thus, the statement is true for x.

21.2 For all positive integers n, we have $1+2+3+..+n = \frac{1}{2}n(n+1)$

Proof: Suppose for the sake contradiction, that the statement is false. Let X be the set of counterexamples. So $X = \{n \in \mathbb{N} : 1+2+...+n \neq \frac{1}{2}n(n+1)$, so $X \neq \emptyset$.

- By **WOP**, X contains the least element.
- $x \neq 1$ because 1 is the same as $1 = \frac{1}{2}(1)(1+1) = 1$, so $x > 1$ ✓
- Then for $x-1$, the proposition is true, so:

$$1+2+...+(x-1) = \frac{1}{2}(x-1)(x-1+1) = \frac{(x-1)\cdot x}{2}$$

Now add x to both sides, so

$$1+2+...+(x-1)+x = \frac{(x-1)\cdot x}{2} + x = \frac{x^2-x+2x}{2} = \frac{x(x+1)}{2} \Rightarrow\Leftarrow$$

Thus, the proposition holds for x, which contradicts the fact that $x \in X$

21.3 Prove that For all $n \in \mathbb{N}$, $n < 2^n$.

- Suppose for the sake of contradiction, the statement is false. Let X be the set of counterexamples. So $X = \{n \in \mathbb{N} : n \not< 2^n\}$. Since the statement is false, $X \neq \emptyset$. By the **WOP**, X contains the smallest element.
- $x \neq 0$ because $0 < 2^0 = 1$, which is true ✓
- $x \neq 1$ because $1 < 2^1 = 2$, which is also true. So $x \geq 2$.
- Consider $x-1 \in \mathbb{N}$, for which the statement is true.

$$(x-1) < 2^{x-1}$$

$$2\cdot(x-1) < 2^x$$

Since $x \geq 2$, $x+x \geq x+2$, then $x \leq 2(x-1)$

$$\frac{x}{x-1} \leq 2$$

Since $x-1 < 2^{x-1}$ and $\frac{x}{x-1} \leq 2$

$$\Rightarrow (x-1)\cdot\frac{x}{x-1} < 2\cdot 2^{x-1} \Rightarrow x < 2^x$$

But $x \in X$ $\Rightarrow\Leftarrow$

4. Prove that $n! \leq n^n, \forall$ positive n.

- Suppose for sake of contradiction, the statement is false. Let X be the set of counterexamples. So $X = \{n \in \mathbb{Z}^+ : n! \not\leq n^n\}$. Since this statement is false, $X \neq \emptyset$. By the **WOP**, X contains the smallest element.
- $x \neq 0$, because $0! = 1 \leq 0^0 = 1$.
- $x \neq 1$, because $1! = 1 \leq 1^1 = 1$, so $x > 1$ and $x-1 > 0$.

- Consider $x - 1 \in \mathbb{Z}^+$, for which the statement is true. Multiply each side by x.

$$(x-1)! < (x-1)^{x-1}$$

$$x \cdot (x-1)! < x \cdot (x-1)^{x-1}$$
$$x! < x \cdot (x-1)^{x-1}$$

Since we know that $x - 1 < x$

$$x! < x \cdot x^{x-1}$$
$$x! < x \cdot x^{x-1} = x^{1+x-1} = x^x \Rightarrow\Leftarrow$$

But this is a contradiction, thus because our supposition of the false statement is false, then the proposition holds true. □

5. Prove that for all positive integers n, we have $1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! = (n+1)! - 1$

- Suppose for sake of contradiction that the statement is false. Let X be the set of counterexamples. So $X = \{n \in \mathbb{Z}^+ : 1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! \neq (n+1)! - 1\}$
Since this statement is false, $X \neq \emptyset$. By the **WOP**, X contains the smallest element.
- $x \neq 0$, because:

$$0 \cdot 0! = (0+0)! - 1 \checkmark$$

- $x \neq 1$, because:

$$1 \cdot 1! = (1+1)! - 1 \checkmark$$

Therefore, $x > 1$ and $x - 1 > 0$.
- Consider $x - 1 \in \mathbb{Z}^+$, for which the statement is true.

$$1 \cdot 1! + 2 \cdot 2! + ... + (x-1) \cdot (x-1)! = ((x-1)+1)! - 1$$

$$1 \cdot 1! + 2 \cdot 2! + ... + (x-1) \cdot (x-1)! = x! - 1$$

Let's add $x \cdot x!$ to both sides

$$1 \cdot 1! + 2 \cdot 2! + ... + (x-1) \cdot (x-1)! + (x \cdot x!) = (1+x) \cdot x! - 1$$

Since $n! = n \cdot (n-1) \cdot ... \cdot 2 \cdot 1$ and $1 + x = x + 1$

$$1 \cdot 1! + 2 \cdot 2! + ... + (x-1) \cdot (x-1)! + (x \cdot x!) = (x+1) \cdot x! - 1 \Rightarrow\Leftarrow$$

- Thus, because our supposition that "the statement is false" is false, then the proposition holds true. □

3 Induction

Principle of mathematical induction:

Let A be a set of natural numbers. If:

1. $0 \in A$ (**Basis Step**)
2. $\forall k \in \mathbb{N}, k \in A \Rightarrow k + 1 \in A$ (**Inductive Hypothesis**), then $A = \mathbb{N}$

- Start by saying what the statement is that you want to prove: "Let $P(n)$ be the statement..." To prove that $P(n)$ is true for all $n \leq 0$, show the following:
  - Prove (or show) that $P(n = 0)$ is true. Just plug in $n = 0$.
  - Inductive Case: Assume that the statement is true for $n = k$, so $P(n = k)$ is true. Show that:
    $\Rightarrow$ $P(n = k + 1)$ is true for all $k \leq 0$.
  - Once both sides are shown to be true, you can conclude, "Therefore, by the principle of mathematical induction, the statement is true for all $n \leq 0$."

3.1 Examples:

1. Prove that for $n \geq 1, 1+2+3+...+n = \frac{n(n+1)}{2}$

Proof: Let $P(n)$ be the statement $1+2+3+...+n = \frac{n(n+1)}{2}$. We will show that $P(n)$ is true for all $n \geq 1$.

- For $n = 1$, $1 = \frac{1(1+1)}{2} = 1$, so $P(1)\checkmark$.
- Assume $P(k)$ (or for $n = k$), the statement is true, $k \geq 1$. Then, we need to show it is true for $P(k+1)$.

$$1+2+3+...+k = \frac{k(k+1)}{2}$$

  is true, then

$$1+2+3+...+k+k+1 = \frac{k(k+1)}{2} + k + 1$$

$$\frac{k(k+2)}{2} + k + 1 = \frac{k(k+1) + 2 \cdot (k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

  The right hand side is true for $n = k+1$.

- Therefore $P(k+1)$ is true, so by induction, $P(n)$ is true $\forall n \geq 1$.

2. Prove that for all $n \in \mathbb{N}$, $6^n - 1$ is a multiple of 5.

- Let $P(n)$ be the statement that "$6^n - 1$ is a multiple of 5." We show that $P(n)$ is true for all $n \in \mathbb{N}$.
  (a) For $n = 0$, $P(0)$ is true: $6^0 - 1 = 0$, which is a multiple of 5.
  (b) Assume $P(k)$ is true for $n = k$, so $6^k - 1$ is a multiple of 5. Then $\exists a \in \mathbb{Z} : 6^k - 1 = 5a$, so $6^k = 5a + 1$. Now we multiply both sides by 6. Then $6 \cdot 6^k = 6 \cdot (5a + 1)$

$$6^{k+1} = 30a + 5 = 6(6a + 1) = 5w$$

  where $w = 6a + 1 \in \mathbb{Z}$, so

$$5 | 6^{k+1} - 1$$

  so $P(k+1)$ is true. Thus, by induction, $P(n)$ is true for all $n \in \mathbb{N}$

3. Prove $10^0 + 10^1 + ... + 10^n < 10^{n+1} \forall n \in \mathbb{N}$

Proof: Assume $P(n)$ is the above statement, so we prove that $P(n)$ is true for all $n \geq 0$. Let $n = 0$, $10^0 = 1 < 10^{0+1} = 10\checkmark$

(a) Assume it is true for $P(k)$, so it $10^0 + ... + 10^k < 10^{k+1}$.
(b) We need to show it is true for $n = k+1$.
(c) Then, consider $10^0 + ... + 10^k + 10^{k+1} < 10^{k+1} + 10^{k+1}$ by adding 10^{k+1} to both sides.

$$10^{k+1} + 10^{k+1} = 2 \cdot 10^{k+1} < 10 \cdot 10^{k+1} = 10^{k+2}$$

  Therefore, it is true for $n = k+1$, thus by induction, $P(n)$ is true for all $n \geq 0$ $\square$

4. Prove that $n! \leq n^n, \forall$ positive n.

(a) Assume $P(n)$ is the above statement, so we prove that $P(n)$ is true for all positive n $n! \leq n^n$. Let $n = 1$, $1! \leq 1^1$ $\checkmark$.
(b) Assume it is true for $P(k)$, so $k! \leq k^k$. If it is true for k, then it must be true for $k+1$, so $(k+1)! \leq (k+1)^{k+1}$.
(c) Then consider adding $(k+1)^{k+1}$ on both sides.