Lecture 8 - Discrete Mafematiks

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1 Proofs By Contradiction:

To prove "If A, then B", assume "If A and NOT B", then carry out until your work reaches a contradiction.

1.1 Examples:

1. If a and b are real numbers and $a \cdot b = 0$, then $a = 0$ or $b = 0$.

   - **Proof By Contradiction:** Let a, b be real numbers, such that $a \cdot b = 0$ and suppose that, for the sake of contradiction, $a, b \neq 0$.
   - Then, since $a, b \neq 0$, then we can divide by a and b.

   $$a \cdot b = 0$$
   $$a \cdot \not{b} \cdot \frac{1}{\not{b}} = 0 \cdot \frac{1}{b}$$
   $$a = 0 \Rightarrow\Leftarrow$$

   - Similarly,

   $$a \cdot b = 0$$
   $$b \cdot \not{a} \cdot \frac{1}{\not{a}} = 0 \cdot \frac{1}{a}$$
   $$b = 0 \Rightarrow\Leftarrow$$

   $\square$

2. Let a be a number with $a > 1$. Prove that $\sqrt{a}$ is strictly between 1 and a. **Note:** *We need to show that* $1 < \sqrt{a} < a$.

   - **Proof By Contradiction:** Assume a is a number, such that $a > 1$ and for the sake of contradiction, $\sqrt{a} \not> \sqrt{1}$. Then, $\sqrt{a} \leq 1$. So squaring both sides, we get $\sqrt{a}^2 \leq 1^2 \Rightarrow a \leq 1$, but $a > 1 \Rightarrow\Leftarrow$
   - Suppose a is a number such that $a > 1$ an for the sake of contradiction $\sqrt{a} \not> a$. Then $\sqrt{a} \leq a$. After squaring, we get $\sqrt{a}^2 \leq a^2$. $a \leq a^2$.
   - Since $a > 1$, we can divide both sides by a, thus $\frac{a}{a} \leq \frac{a^2}{a} \Rightarrow 1 \leq a \Rightarrow\Leftarrow \square$

3. Suppose n is an integer divisible by 4. Then $n + 2$ is not divisible by 4.

   - **Proof By Contradiction:** Suppose $n \in \mathbb{Z} : 4|n$, and for the sake of contradiction, $4|n + 2$. Then $\exists a, b \in \mathbb{Z} : n = 4a$ and $n + 2 = 4b$.

   $$4a + 2 = 4b$$
   $$2 = 4b - 4a$$
   $$2 = 4(a - b)$$
   $$\frac{2}{4} = b - a$$

   - However, since $b, a \in \mathbb{Z}$, $b - a$ must also be an integer, but $\frac{2}{4}$ is a rational number $\Rightarrow\Leftarrow \square$

4. Prove by contradiction that consecutive integers can not be both even.

   - **Proof:** Let $x \in \mathbb{Z}$, then for the sake of contradiction, $2|x$ and $2|x + 1$.

- Then $\exists a \in \mathbb{Z} : x = 2a$ and $\exists b \in \mathbb{Z} : x + 1 = 2b$. So $x + 1 = 2a + 1 = 2b$, so

$$1 = 2b - 2a$$

$$1 = 2(b - a)$$

$$\frac{1}{2} = b - a \Rightarrow\Leftarrow$$

- $b - a$ is an integer, but $\frac{1}{2} \in \mathbb{Q}$ $\square$

2 Proof By Contrapositive:

Assume NOT B, then NOT A is implied.

2.1 Examples:

1. If x is odd, then x^2 is odd.
   If x^2 is not odd, then x is not odd.
2. If the battery is fully charged, the car will start.
   The car will not start if the battery is not fully charged.
3. If A or B, then C.
   If not C, then not A and not B

3 Anagrams

3.1 Examples:

1. How many different anagrams including nonsensical words can be made from "FACETIOUSLY" if we require all six vowels must remain in alphabetical order, but not contiguous with each other?
   - In how many ways can 11 letters be arranged?
     11!
   - In how many ways can 6 vowels be permuted?
     6!
   - Therefore, the number of options for generating arrangements with the specific order of vowels is $\frac{11!}{6!}$
2. "SUCCESS" If we require that the first and the last letters must be "S".
   - There are 5! ways to permute the inner 5 letters. However, there are 2 "C"'s.
   - $\frac{5!}{2!}$
3. 20 people are to be divided into two teams with ten players in each. How many ways could this be done?
   - $\frac{20!}{10!^2 \cdot 2!}$