Lecture 7 - Discrete Mafematiks

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1 Partitions

Definition: Let A be a set. A partition of A is a set of non-empty, pairwise disjoint sets whose union is A.

- A partition is a set of sets, where sets are subsets of A
- The members of the partitions are called parts.
- Parts of a partition are nonempty
- Parts of a partition are mutually disjoint

1.1 Examples:

1. **Proposition:** Let $A = \{1, 2, 3, 4, 5, 6\}$
   - $P_1 = \{\{1, 2\}, \{3\}, \{4, 5\}, \{6\}\}$
   - $P_2 = \{\{1, 2, 3, 4, 5, 6\}\}$
   - $\{\{1\}, \{2\}, \{3\}, \{5\}, \{4\}, \{6\}\}$
   - P is an equivalence relation on A defined as $\forall x, y \in P$.
   - $R \equiv P$ so that $x \equiv y$, as *being-in-the-same-part-as*, then the equivalence class

   $$[y] = \{\forall x \in P : x \equiv y\}$$

2. **Proposition:** Let A be a set and let P be a partition on A. The relation P is an equivalence relation on A.
   - **Proof:**
   - **Reflexive:** Let x be any element in A. Then $x \in P$ since $P = \cup_{i=1}^{n} P_i = A$, so X must belong to some part of P, so $x \in P_i$ for $i = 1, ..., n$. Then xPx, so $x \in P_i$ and $x \in P$. Thus, P is reflexive.
   - **Symmetric:** Suppose xPy for some $x, y \in A$. Then $x \in P_i$ and $y \in P_i$ for some $i = 1, 2, ..., h$. So x and y belong to the same part of a partition P. $x \in P$ and $y \in P$, so $xPy \Rightarrow yPx$, so P is symmetric.
   - **Transitive:** Let $x, y, z \in A : xPy$ and yPz. Since xPy, then $\exists P_i$ so that $x \in P_i$ and $y \in P_i$, so both $x, y \in P$. Similarly, yPz, so $y \in P_j$ and $z \in P_j$, so $y \in P$ and $z \in P$. It follows that P_j must be the same as P_i due to the definition of a partition (*all parts must be mutually disjoint*). This implies that P_i is the same as P_j, so xPz because $z \in P_i$.

3. **Proposition:** Let P be a partition on A, and let P be the *in-the-same-part-as* relation on A. Then the equivalence classes of P are the parts of a partition.
   Need to Show:
   (1) How many classes are induced by P?
   (2) How many elements in each equivalence class are there?
   (1) Depends on the cardinality of a set A and how many elements in each part.
   Let R be an equivalence relation on a finite set A. If all the equivalence classes of R have the same number of element in it; If each equivalence class consists of of m elements, then there are $\frac{|A|}{m}$ equivalence classes.

4. **Example:** Consider a word "HELLO". $A = \{H, E, L, L, O\}$. How many equivalence classes can be formed consisting of 2 elements. Where equivalence relations are derived as rearranging letters.

- So there are 5! ways of rearranging letters in a word "HELLO", and since each equivalence class will consist of only 2 elements, then the number of equivalence classes is $\frac{5!}{2} = \frac{120}{2} = 60$ different equivalence classes.

5. **Example:** Let $A = 2^{\{1,2,3,4\}}$ power set of a set $\{1, 2, 3, 4\}$. R has the *same-size-as* relation. This relation partitions A into 5 parts.

   - $|A| = 2^4 = 16$
   - Equivalence Classes:
     $[\emptyset] = 1$
     $[\{1\}] = 4$
     $[\{1, 2\}] = 6$
     $[\{1, 2, 3\}] = 4$
     $[\{1, 2, 3, 4\}] = 1$

2 Proof by Contrapositive:

- Consider the statement: "If A, then B". Then contrapositive of this statement implies "If NOT B then NOT A".
- Assume NOT B and work to prove NOT A.

2.1 Examples:

1. **Example:** Let R be an equivalence relation on set A and let $a, b \in A$. If $a \not R b$, then $[a] \cap [b] = \emptyset$.

   - **Proof:** Suppose $[a] \cap [b] \neq \emptyset$ for some $a, b \in A$.
   - Then there is some $x \in [a] \cap [b]$, so $x \in [a]$ and $x \in [b]$. Therefore, xRa and xRb since R is an equivalence relation, it is **symmetric**, so $xRa = aRx$ and **transitive** due to the transitive property $xRb \Rightarrow aRb$.

3 Prove by Contradiction:

To prove "If A, then B".
Assume A for the sake of contradiction NOT B.
Argue until we reach a contradiction.

3.1 Examples:

1. **Example:** Prove by Contradiction. If x is a real number, then x^2 is NOT negative.

   - **Proof:** Suppose for the sake of contradiction there exists a real number so that $x^2 < 0$.
   - However, x can be positive, zero, or negative.
     - If x is positive, so $x > 0$. x^2 must also be greater than 0 because $x \cdot x > 0 \cdot x$.
     - If x is 0, so $x = 0$. x^2 is also 0 because $x \cdot x = 0 \cdot x$.
     - If x is negative, so $x < 0$. x^2 is greater than 0 because $-x \cdot -x > -x \cdot 0 \Rightarrow\Leftarrow$