Lecture 6 - Discrete Mafematiks

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July 16, 2020

1 Relations

We know that a relation is a set of ordered pairs.

1. If for all $x \in A$ we have xRx, then R is called **reflexive**.
2. If for all $x \in A$ we have $x \not{R} x$, then R is called **irreflexive**.
3. If for all $x, y \in A$ we have $xRy \Rightarrow yRx$, then R is **symmetric**.
4. If for all $x, y \in A$ we have $(xRy \wedge yRx \Rightarrow x = y$, then R is **antisymmetric**.
5. If for all $x, y, z \in A$ we have $(xRy \wedge yRz) \Rightarrow xRz$, then R is **transitive**.

Note: *Relations which are reflexive, symmetric, and transitive are called* **equivalent relations**.

Remark: *Relations can be represented by directed graphs, digraphs, or matrices.* (see (1),(2),(3))

Definition: A **Directed graph** or digraph is a diagram with vertices (**def:** *elements in a set*) and edges (**def:** *represent if elements if are related*). **Definition:** The **adjacency matrix** is also called the **connection matrix**, a_{ij} are the set of non-negative integers, where a_{ij} = the number of arrows coming out of the corresponding vertices. **Remark:** loops **do** count (see (4),(5)).

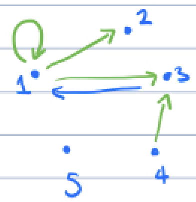
Definition: The **undirected graph** occurs when we don't have arrows.

1.1 Relations Examples:

① $A = \{1,2,3,4,5\}$

$R = \{(1,1),(1,2),$
$(1,3),(4,3),$
$(3,1)\}$

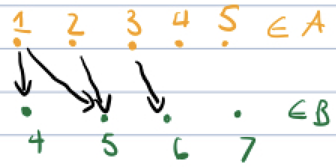
Resulting Graph →



② $A = \{1,2,3,4,5\}$

$B = \{4,5,6,7\}$

Resulting Graph →



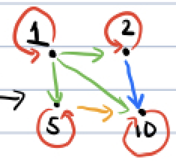
Let $R = \{(1,4),(1,5),(2,5),(3,6)\}$

③ Let $A := \{a \in \mathbb{N} : a \mid 10\}$ and let R be the relation "|" restricted to A. Draw the Digraph

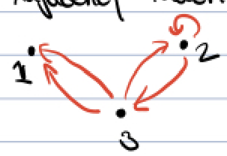
$A = \{1,2,5,10\}$

$R = \{(1,1),(1,2),(1,10),(1,5),$
$(2,2),(2,10),$
$(5,5),(5,10),$
$(10,10)\}$

Resulting Graph →



④ Find Adjacency Matrix.

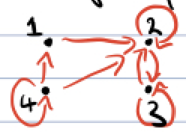


Resulting Matrix →

$M =$

	1	2	3
1	0	0	0
2	0	1	1
3	2	1	0

⑤ Find Adjacency Matrix.

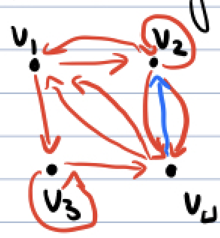


	1	2	3	4
1	0	1	0	0
2	0	1	1	0
3	0	1	0	0
4	1	1	0	1

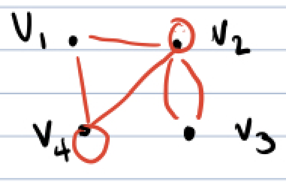
⑥ Given a Matrix, draw a digraph.

$A =$

	V_1	V_2	V_3	V_4
V_1	0	1	1	0
V_2	1	1	0	2
V_3	0	0	1	1
V_4	2	1	0	0



⑦ The adjacency Matrix of G is the matrix $A = (a_{ij})$ over the set of non-negative integers, s.t. a_{ij} = the num of edges connecting V_i and V_j.



$M =$

	V_1	V_2	V_3	V_4
V_1	0	1	0	1
V_2	1	1	2	0
V_3	0	2	0	0
V_4	1	1	0	1

1.2 Equivalence Relations

An equivalence relation is the one which is reflexive, symmetric, and transitive. In order to show that R is an equivalence relation, simply prove refelxivity, symmetry, and transitivity.

Congruence modulo n

1. Let n be a positive integer. We say that x and y are congruent modulo n and $x \equiv y \ (mod\ n)$ if $n|(x-y)$. n must be a factor of the difference between x and y.
2. $3 \equiv 14 \ (mod\ 5) \Rightarrow 3 - 13 = -10$, which is divisible by 5.
3. $2 \equiv 2 \ (mod\ 5) \Rightarrow 2 - 2 = 0$, which is a divisible by 5.
4. Show congruence modulo n is an equivalence relation. Thus, we show it satisfies the following:
   - **Reflexivity:** Consider $x \in \mathbb{Z}$, so that $x \equiv x \ (mod\ n)$ since $n|(x-x)$, $n|0$, so $x \equiv x$
   - **Symmetry:** Let x and y be integers, so that $x \equiv y \ (mod\ n)$, so $n|x-y$, so $\exists a \in \mathbb{Z} : x - y = n \cdot a$. However, $-(y-x) = c \cdot a$, so $y - x = n(-a)$, so $n|y-x$, so $y \equiv x \ (mod\ n)$, so it is symmetric.
   - **Transitivity:** Consider $x, y, z \in \mathbb{Z}$, so that $x \equiv y \ (mod\ n)$ and $y \equiv z \ (mod\ n)$, and $n|x-y$ and $n|y-z$. Thus, $\exists k, m \in \mathbb{Z}$, such that $x - y = n \cdot k$ and $y - z = n \cdot m$. We can add them together, where we have $(x-y)+(y-z) = nk + nm = n(k+m)$, so $\exists w \in \mathbb{Z} : w = k + m$, so $x - z = n \cdot w$, so $n|x-z$, so $x \equiv z \ (mod\ n)$

1.3 Equivalence Classes

Definition: Let R be an equivalence relation established on set A, and $a \in A$. The equivalence class of a, denoted $[a]$, is the subsets of A so that:

$$[a] = \{x \in A : xRa\}$$

Equivalence classes are mutually disjoint subsets, so the union of all equivalence classes is the entire set of A.

Remark: We could say that A is partitioned into disjoint equivalence classes.

- Say we have $a, b, c, ..., x \in A$. $[a] \cup [b] \cup [c] \cup ... \cup [x] = A$ **or** $\cup_{i=1}^{n}[a_i] = A$, **while** $\cap_{i=1}^{n}[a_i] = \emptyset$

1.4 Equivalence Class Examples:

1. Consider equivalence relation $mod\ 2$ on the set of all integers. Find all equivalence classes.
   - Consider $x \in \mathbb{Z}$, so that $x \equiv 0 \ (mod\ 2)$. So $2|(x-0)$, so $\exists k \in \mathbb{Z} : x = 2k$ is even. So, it is a subset of all even integers. So, the equivalence class is $[0]$.
   - Consider $x \in \mathbb{Z}$, so that $x \equiv 1 \ (mod\ 2)$, so $2|x-1$, so $\exists k \in \mathbb{Z} : x - 1 = 2k$, so $x = 2k+1$, which is odd. So an equivalence class is $[1] =$ the set of all odd integers.
2. Let $A = \{1, 2, 3, 4\}$. Find $[1]$ if $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$
   - $[1] = \{1, 2\}$
3. Consider a set $A = \{x \in \mathbb{Z} : 100 < x < 200\}$ Let R be an equivalence relation that has the same digit on the set A. Find $[123]$.
   - $[123] = \{120, 121, 122, 123, 124, 125, 126, 127, 128, 129\}$
4. R has the same szie on the set $2^{\{1,2,3,4,5\}}$. Find $[\{1,3\}]$.
   - $[\{1,2\}] = \{\{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}$

Note: *we are comparing on the basis of cardinality*