Lecture 5 - Discrete Mafematiks

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1 Combinatorial Proof

In order to prove by **combinatorial proof**, we need to find a question which can be answered by the left-had side of a given equation as well as the right-had side. If that question is answered via both approaches, then that would be the proof.
Note: *The challenging part is to come up with the question or situation, for which you need to find the answer or explanation.*

1.1 Examples:

(1) **Proposition:** Let A and B be finite sets. Then $|A| + |B| = |A \cup B| + |A \cap B|$

- $|A| + |B|$ $(lhs) = |A \cup B| + |A \cap B|$ (rhs) where A and B are finite sets.

→ **Combinatorial Proof:** Imagine we assign labels to every object in these sets. *(now we ask the question)*

Question: How many labels have we assigned?

- Since the cardnialities of sets A and B are $|A|$ and $|B|$. respectively, then the total number of labels is $|A| + |B|$
- On the other hand, we have assigned at least one label to the elements in $A \cup B$, so the number of labels in $|A \cup B|$. However, elements in the intersection $A \cap B$ receive 2 labels because they are included in $|A \cup B|$.
- Therefore, the total number of labels distributed is $|A \cup B| + |A \cap B|$ (rhs).
- Since both sides of the statement answer the same question, $|A| + |B| = |A \cup B| + |A \cap B|$ □

(2) **Proposition:** Let n be a positive integer. Then $2^0 + 2^1 + 2^2 + ... + 2^{n-1}$ $(lhs) = 2^n - 1$ (rhs) □

→ **Combinatorial Proof:** Consider a set $N = \{1, 2, 3, ..., n\}$*(now we ask the question)*

Question: How non-empty subsets does N have?

- The power set of N has 2^n subsets, but $\emptyset$ is a subset of a power set. Thus, 2^{n-1} is the total number of non-empty subsets.
- Now, consider subsets with the largest element in that given subset.
- Let k be the largest element in that given subset. Therefore, the largest k can be is n and the smallest is 1, so subsets must look like $\{..., k\}$, so the largest element will be $k - 1$ other options for other elements.
- Therefore, 2^{k-1} subsets will look like $\{..., k - 1\}$, where the largest element is $k \leq n$.
- Thus, the number of options is $2^0 + 2^1 + 2^2 + ... + 2^{n-1}$ for non-empty subsets.□

(3) **Proposition:** $1 + 2 + 3 + ... + n = \binom{n+1}{2}$

→ **Combinatorial Proof:** Consider a set $A = \{1, 2, 3, ..., n + 1\}$ elements. *(now we ask the question)*

Question: How many subsets of A contain exactly 2 elements?

- We need to choose 2 elements out of $n+1$ choices, so there are $\binom{n+1}{2}$ possible subsets with 2 elements.
- Now consider subsets where one element is greater than another.
- Assume the larger element is 2. Then there is 1 choice. Now assume the larger element is 3, then we have 2 choices. ... The larger element is $n + 1$, then there are n choices.

- Since each two element subset must be exactly one of these cases, the total number of possibilities is the sum of all of thee choices, which is the left-hand-side.
- Therefore, we obtain both explanations to the same question, so the statement is true.$\square$

(4) **Proposition:** Prove the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

→ **Combinatorial Proof:** Consider a set that consists of all elements $A = \{1, 2, 3, ..., n\}$ *(now we ask the question)*

Question: How many subsets of A of A can contain k elements?

- We need to choose k elements out of n choices, so there are $\binom{n}{k}$ possible subsets with k elements.
- Now consider subsets where one element is greater than another.
- Consider how many subsets where exactly 1 element (the largest element) is removed.
- There will only be $n-1$ objects to choose from out of the n objects.
- Consider subsets which do not contain the largest element, n. Now we have $n-1$ options for choosing k elements.
- So we will have $\binom{n-1}{k}$ However, the remaining subsets of $k-1$ elements can be chosen from $n-1$ choices and containing the largest element so there are $\binom{n-1}{k-1} + \binom{n-1}{k}$ options. $\square$

2 Relations

Definition: A relation is a set of ordered pairs.

- **ex:** $R = \{(1, 2), (2, 3), (4, 5)\}$
- **Notation:** $(x, y) \in R \Rightarrow xRy$
- **Remark:** R is the criteria with respect to which x and y are related. It can be "equal to", "less than/greater than" or "less than or equal to/greater than or equal to."

Relations can be established between sets.

- Consider the sets A and B and the relations "$\subseteq$". We say R in a relation is from A to B provided $R \subseteq A \times B$
- **ex:** Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$
- $R = \{(1, 1,), (2, 2), (3, 3), (4, 4)\}$ relation on set A with equivalency.
- $S = \{(1, 2), (3, 2)\}$ another relation on A

Definition: Let R be a relation. The intverse of R is R^{-1} is the relation. Reversing x and y. Thus, if $(x, y) \in R$ and xRy, then $yR^{-1}x$.

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

- $R = \{(1, 5), (2, 3), (7, 12)\}$, so R^{-1} would be the inverse.
- If R is a relation on set A, then R^{-1} is an inverse relation on set A

2.1 Properties of Relations:

1. If for all $x \in A$ we have xRx, then R is called **reflexive**.
2. If for all $x \in A$ we have $x \not R x$, then R is called **irreflexive**.
3. If for all $x, y \in A$ we have $xRy \Rightarrow yRx$, then R is **symmetric**.
4. If for all $x, y \in A$ we have $(xRy \wedge yRx \Rightarrow x = y$, then R is **antisymmetric**.
5. If for all $x, y, z \in A$ we have $(xRy \wedge yRz) \Rightarrow xRz$, then R is **transitive**.

Note: *Relations which are reflexive, symmetric, and transitive are called* **equivalent relations**.

2.2 Examples:

1. Consider the set $\mathbb{Z}$ with the relation "=".
   - This is a reflexive relation because we know that $x = x$.
   - This is symmetric because $\forall x, y \in \mathbb{Z}$, if $x = y \Rightarrow y = x$
   - This is antisymmetric because $\forall x, y \in \mathbb{Z}, (x = y \wedge y = x) \Rightarrow x = y$
   - This is not irreflexive because $\forall x \in \mathbb{Z}, x = x$, which not $x \neq x$. This is **false** $\forall x \in \mathbb{Z}$. The $=$ relation is not irreflexive.
2. Consider a relation "$\leq$" on $\mathbb{Z}$.
   - It is reflexive because we know that $x \leq x$.
   - This is not symmetric because $x \leq y \Rightarrow y \leq x$ is not necessarily true.
   - This is transitive because $(x \leq y \wedge y \leq Z) \Rightarrow x \leq z$. This is antisymmetric because $\forall x, y \in \mathbb{Z}, (x \leq y \wedge y \leq x) \Rightarrow x = y$
3. Consider a relation "$<$" on $\mathbb{Z}$
   - $\forall x \in \mathbb{Z}, x < x$ False, so not reflexive.
   - $\forall x, y \in \mathbb{Z}$, so not symmetric.
   - $\forall x, y, z \in \mathbb{Z}, (x < y \wedge y < x) \Rightarrow (x < z)$ so it is reflexive.
   - $\forall x \in \mathbb{Z}, x < x$, so it is not irrelfexive.
   - $\forall x, y \in \mathbb{Z}$, $(x < y) \wedge (y < x) \Rightarrow x = y$ is antisymmetric because it is **vacuously true**.
4. Consider the relation $A = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$
   - This is reflexive. $\forall x \in A, xRx$
   - This is not irreflexive. $\forall x \in A, xRx$
   - This is antisymmetric. $\forall x, y \in A$, $(x = y) \wedge (y = x) \Rightarrow x = y$
   - This is symmetric. $\forall x, y \in A, xRy \Rightarrow yRx,\ so\ x = y \Rightarrow y = x$
   - This is transitive. $\forall x, y, z \in A,\ then\ (x = y \wedge y = z) \Rightarrow x = z$