Lecture 4 - Discrete Mafematiks

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1 Quantifiers

Definition: Phrases which can be symbolically represented in proofs and those are symbols called quantifiers.

$\exists$ "there exists" ... this is an existential qunantifier

$\forall$ "for all" ... this is a universal quantifier

$\exists!$ "there exists a unique"

$\neg$ "not" ... negating quantifier

: "such that"

1.1 Proving Existential Statements:

- To prove that $\exists x \in A$, assertions about x.
  - Let x be *give an explicit example ... here we use a definition of being odd, even, or composite* then show that x satisfies all assertions.
  - Therefore, x satisfies the required assertions.

1.2 Proving Universal Statements:

- prove $\forall x \in A$, assertions about x
  - Let x be any element of A, show that x satisfies the assertions using only the fact that $x \in A$ and no further assumptions.
  - Therefore x satisfies the required assertions

1.3 Negating Quantified Statements:

- $\neg(\forall x \in \mathbb{Z} x$ is prime$) \rightarrow$ "Not all integers are prime"
- $\neg(\exists x \in A$, assertions about $x) \rightarrow$ "None of the elements of A satisfies the assertions about x."
- $\neg(\forall x \in A$, assertions about $x) \rightarrow$ "Not all of the elements of A satisfies the assertions about x."

1.4 Examples:

(1) $\exists x \in \mathbb{N} : 2|x$

(2) $\forall x \in \mathbb{Z} : x$ is a prime number.

(3) (1.1) $\exists x \in \mathbb{Z} : x$ is even and prime.

- Let $x = 2$, which is even and prime $\square$

(4) (1.2) Let $A = \{x \in \mathbb{Z} : 6|x$. Prove the statement that $\forall x \in A$,x is even.

- Let $x \in A$. Then $\exists y \in \mathbb{Z} : x = 6y = (2 \cdot 3)y = 2(3y)$ where $2|x$, therefore x is even $\square$

(5) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : x + y = 0$. True

Proof: Let $x \in \mathbb{Z}$. Consider $y \in \mathbb{Z} : y = -x$.

- Then $x + y = x + (-x) = 0\square$

(6) $\exists y \in \mathbb{Z} : \forall x \in \mathbb{Z}, x + y = 0$ False.

Proof: Let $y \in \mathbb{Z}$. Consider $x_1 \in$

(7) $\exists! y \in \mathbb{Z} : \forall x \in \mathbb{Z}, x + y = x$. True.

Proof: Lets assume that there are $y = 0$. $y \exists \mathbb{Z}$ and $y* \in \mathbb{Z} : \forall x \in \mathbb{Z}, x + y = x \ and \ x + y* = x$.

- Since $x + y = x + y*$, $y = y*$

(8) $\forall x \in \mathbb{Z} : x \ is \ odd.$

- $\exists x \in \mathbb{Z} : \neg(x \ is \ odd)$

(9) $\exists x \in \mathbb{Z} : x = x + 1$

- $\forall x \in \mathbb{Z} : x \neq x + 1$

1.5 More Examples:

True/False Statements:

(1) $\forall x, \forall y, x + y = 0$ F

(2) $\forall x, \exists y, x + y = 0$ T

(3) $\exists x, \forall y, x + y = 0$ F

(4) $\exists x, \exists y, x + y = 0$ T

(5) $\forall x, \forall y, xy = 0$ F

T (6) $\forall x, \exists y, xy = 0$

T (7) $\exists x, \forall y, xy = 0$

T (8) $\exists x, \exists y, xy = 0$.

True or False?

(1) $\exists! x \in \mathbb{N} : x^2 = 4$ T

(2) $\exists! x \in \mathbb{Z} : x^2 = 4$ F

x can be 2 and x can be -2, therefore false

(3) $\exists! x \in \mathbb{R} : x^2 = 3$ F

It is not unique because it can be x = +/- sqrt 3

(4) $\exists! x \in \mathbb{Z}, \forall y \in \mathbb{Z} : xy = x$ T

Negate the following:

(3) $\exists x \in \mathbb{N} : x > 0$

- $\forall x \in \mathbb{N}, x \not> 0$

(4) $\forall x \in \mathbb{N} : x + x = 2x$.

- $\exists x \in \mathbb{N} : x + x \neq 0$

(5) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x > y$

- $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \not> y$

2 Operations

"$\cup$" The "Union", so $x \in A \cup B$ can either be in set A or set B, and this holds true.

"$\cap$" The "Intersection", so $x \in A$ and $x \in B$.

"-" **Set Difference:** Let A and B be ets. The set difference $A - B$, is the set of all elements of A that are not in B.

- $A - B = \{x : x \in A \; and \; x \notin B\}$

"Δ" The "Symmetric difference" of A and B is the set of all elements in A, but not in B, or vice versa.

Disjoint: If there intersection is an emptyset, so $A \cap B = 0$. If $A_1, A_2, ..., A_k$ is a collection of k sets, then these sets are called pairwise disjoint if intersections of any two sets are empty sets. $A_i \cap A_j = \emptyset, \forall A_i = 1, 2, ..., k$, and $A_j = 1, 2, ..., k$.

2.1 Examples:

(1) **Base:** Suppose $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$.

(a) $A \cup B = \{1,2,3,4,5,6\}$
(b) $A \cap B = \{3,4\}$
(c) $A - B = \{1,2\}$
(d) $B - A = \{5,6\}$
(e) $A \Delta B = \{1,2,5,6\}$

(2) Let A, B, and C be sets. Then:

(a) $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

- **Proof:** $A \cup B \subseteq B \cup A$
- Let $x \in A \cup B$. Then $(x \in A) \wedge (x \in B) = (x \in B) \wedge (x \in A)$
- Let $x \in B \cup A$. Then $(x \in B) \wedge (x \in A) = (x \in A) \wedge (x \in B)$

(b) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

(3) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$

- **Proof:** $A \cup \emptyset = A$
- Then $(x \in A) \wedge (x \in \emptyset)$. However, $x \notin \emptyset$ since by the definition, $\emptyset$ has no elements in it, so $x \in A$ only.
- Let $x \in A$, so $(x \in A) \wedge (x \in \emptyset)$, so $x \in A \cup \emptyset$. Thus, $A \subseteq A \cup \emptyset$, therefore $A \in A \cup \emptyset$ $\square$

(4) $A \cap (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) =$

REVISIT $A \Delta B = (A \cup B) - (A \cap B)$

- **Proof:** Let $x \in A \Delta B$.
- Then, $x \in (A - B) \cup (B - A)$.
- This means that $x \in A - B$ or $x \in B - A$.
  (a) Let $x \in A - B$, so $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in A \cup B$. Since $x \notin B \; so \; x \notin A \cap B$ Therefore, $x \in (A \cup B) - (A \cap B)$
  (a) If $x \in B - A$, so $x \in B$ and $x \notin A$. Since $x \in B$, we have $x \in B \cup A$. Since $x \notin A \; so \; x \notin B \cap A$ Therefore, $x \in (B \cup A) - (B \cap A)$, and $A \Delta B \subseteq (A \cup B) - (A \cap B)$

- Let $x \in (A \cup B) - (A \cup B)$. Then $x \in A \cup B$ and $x \notin A \cap B$.

  Therefore, x is in A, or x is in B, but not in both. Thus, x is either in $B - A$ OR x is in $A - B$, so $x \in (B - A) \cup (A - B)$

  Therefore, $x \in B\Delta A$, so $(A \cup B) - (A \cap B) \subseteq A\Delta B$

Remark: De'Morgan's Law:

  - Let A, B, C be sets.
    Then $A - (B \cup C) = (A - B) \cap (A - C)$ and $A - (B \cap C) = (A - B) \cup (A - C)$

3 Cartesian Product:

Definition: Let A and B be sets. The Cartesian product of A and B is the set of ordered pairs
$A \times B = \{(a, b) : a \in A \; and \; b \in B\}$ $A \times B \neq B \times A$

3.1 Examples:

1. Let $A = \{1, 2, 3\} \; and \; B = \{3, 4, 5\}$
   - $A \times B = \{(1, 3), (1, 3), (1, 5), ..., (3, 4), (3, 5)$
   - $B \times A = \{(3, 1), (3, 2), ..., (5, 2), (5, 3)\}$