Lecture 12 - Discrete Mafematiks

Kori Vernon

August 4, 2020

1 Assorted Notation

"Big Oh" - Left f and g be real-valued functions defined by the set of natural numbers, so $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$. We say that $f(n)$ is $O(g(n))$ provided there exists a positive number M, such that with at most finite exceptions. $|f(n)| \leq M \cdot |g(n)|$ (bound function).
Then, $f(n) = O(g(n))$

"Big Ω" = Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$. Then we say $f(n)$ is $\Omega(g(n))$ provided there is a positive number M such that, with at most finitely many exceptions.

$$|f(n)| \leq M|g(n)|$$

"Big Θ": Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$. We say that $f(n)$ is in $\Theta(g(n))$ provided there are positive number A and B, such that with at most finite number of exceptions:

$$A|g(n)| \leq |f(n)| \leq B|g(n)|$$

"Little Oh": = Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$. Then we say that $f(n)$ is $o(g(n))$ provided

$$lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

1.1 Examples:

1. Consider $f(n) = \frac{n(n-1)}{2}$. Find $O(g(n))$.

$$\frac{|n(n-1)|}{|2|} = \frac{|n^2 - n|}{|2|} = \frac{|n^2|}{|2|} = \frac{1}{2}n^2; M = \frac{1}{2}$$

   Thus, $f(n)$ is $O(n^2)$

2. Consider $f(n) = 4n^5 - \frac{n(n+1)\cdot(n+2)}{3} + 3n^2 - 12$
   Thus, $f(n) = O(n^5)$

3. Find bound and $O(f(n))$

$$f(n) = \frac{n(n+5)}{2}$$

$$\frac{n(n+5)}{2} div\, n^2 = \frac{n^2}{2} + \frac{5n}{2} \leq \frac{1}{2} + \frac{5}{2\cdot 1} = \frac{6}{2} = 3; M = 3$$

   $O(f(n)) = O(n^2)$

4. **Proposition:** Then $f(n)$ is $O(g(n))$ if and only if $g(n)$ is $\Omega(f(n))$.
   - **Proof:** Suppose $f(n)$ is $O(g(n))$. Then $\exists M \in \mathbb{R} : |f(n)| \leq M|g(n)| \forall n \in \mathbb{N}$, but with finitely many exceptions.
   - Then $\frac{1}{M}|f(n)| \leq |g(n)|$, so let $k = \frac{1}{M}$, then

$$|g(n)| \leq k|f(n)|$$

     so $g(n)$ is $\Omega(f(n))$

- $\Leftarrow$ Suppose $g(n)$ is $O(f(n))$, then $\exists M \in \mathbb{R} : |g(n)| \leq M|f(n)|$ with a finite number of exceptions.
- Then, dividing both sides by M gives us $\frac{1}{M}|g(n)| \leq |f(n)|$.
- Let $k = \frac{1}{M}$, so $|f(n)| \geq k|g(n)|$. Thus $g(n)$ is $O(f(n))$

5. **Proposition:** Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$. Then $f(n)$ is $\Theta(g(n))$ if and only if $O(g(n))$ and $f(n)$ is $\Omega(g(n))$

   - **Proof:** Suppose $f(n)$ is $\Theta(g(n))$. Then $\exists A, B \in \mathbb{R}$ such that:

   $$A|g(n)| \leq |f(n)| \leq B|g(n)|$$

   with finitely many exceptions. Because $|f(n)| \leq B|g(n)|$, then $f(n)$ is $O(g(n))$, and because $|f(n)| \geq A|g(n)|$, then $f(n)$ is $\Omega(g(n))$ for finitely many exceptions.
   - $\Leftarrow$ $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.
   - Then $\exists A, B \in \mathbb{R} : A|g(n)| \leq |f(n)|$ for all n with finitely many exceptions.
   - Thus, $A|g(n) \leq |f(n)| \leq B|g(n)|$. Therefore, $f(n)$ is $\Theta(g(n))$.

6. Let $f(n) = \sqrt{n}$, then $lim_{n\to\infty} \frac{\sqrt{n}}{n} = lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so $f(n) = n$ is $o(n)$.

2 Triangle Inequality:

Let a and b be real numbers. Then

$$|a + b| \leq |a| + |b|$$

Proof: (4 cases)

- If neither a nor b is negative, then we have

  $$|a + b| = a + b = |a| + |b|$$

- If $a \geq 0$, and $b < 0$, then $|a| + |b| = a + (-b) = a - b$
  If $|a + b| = a + b$ (when $a + b \geq 0$), and we have

  $$|a + b| = a + b < a < a - b = |a + b|$$

  or

  $$|a + b| = -(a + b)$$

  because $|a| < |b|$, so $a + b < 0$

  then

  $$|a + b| = -(a + b) = -a - b = -|a| + |b| < |a| + |b|$$

  Thus, in both cases, $|a + b| < |a| + |b|$,

- Similarly, when $a < 0$ and $b \geq 0$.

- If both a and b are negative, $a < 0$ and $b < 0$,

  $$|a + b| = -(a + b) = (-a) + (-b) = | - a| + | - b| = |a| + |b|$$

  Thus, $|a + b| \leq |a| + |b|$

3 Floor and Ceiling:

Let $x \in \mathbb{R}$. The floor of x, denoted $\lfloor x \rfloor$, is the largest integer n, such that $n \leq x$. The ceiling of x, denoted $\lceil x \rceil$, is the smallest integer n such that $n \leq x$

3.1 Examples:

1. $\lfloor 2.8 \rfloor = 2$
2. $\lfloor 9.2 \rfloor = 9$
3. $\lceil 2.8 \rceil = 3$

4 Permutations

Definition: Let A be a set. A permutation on a set A is a bijection from A to itself.

Every permutation of a finite set can be expressed as a collection of pairwise disjoint cycles. Furthermore, this representation is unique up to rearranging the cycles and the cycle order of elements within cycles.
Recall: A pairwise disjoint cycle is when any two cycles have no common elements.

Note: A permutation $\tau \in S_n$ is called a **transposition** provided

- There exist $i, j, \in \{1, 2, ..., n\}$ with $i \neq j$, so that $\tau(i) = j$ and $\tau(j) = i$
- for all $k \in \{1, 2, ..., n\}$ with $k \neq i$ and $k \neq j$, we have $\tau(k) = k$

Permutations can be rewritten in terms of transpositions

4.1 Examples:

1. Consider $A = \{1, 2, 3, 4, 5\}$ and $f : A \to A$ is defined as $f = \{(1, 2), (2, 4), (3, 1), (4, 3), (5, 5)\}$, so there is a bijection and rearranged list