Lecture 11 - Discrete Mafematiks

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1 Functions

Proving a function is one-to-one

- **Direct method:** Suppose $f(x) = f(y)$, therefore, $x = y$, thus f is one-to-one
- **Contrapositive method:** Suppose that $x \neq y$ Therefore, $f(x) \neq f(y)$, thus f is one-to-one
- **Contradiction method:** Suppose $f(x) = f(y)$ but $x \neq y \Rightarrow\Leftarrow$, thus f is one-to-one.

Proving a function is onto.

- **Direct method:** Let b be an arbitrary element of B, then construct an element $a \in A : f(a) = b$, therefore f is onto.
- **Set method:** Show that sets $im\ f = B$.

1.1 Examples:

1. $f : \mathbb{Z} \to \mathbb{Z}, f(x) = 3x + 4$. Show that f is one-to-one.
   - **Proof by direct method:** Suppose $f(x) = f(y)$, then

$$3x + 4 = 3y + 4$$

$$3x = 3y \Rightarrow x = y$$

2. Let $f : \mathbb{Z} \to \mathbb{Z}$, defined by $f(x) = |x|$. Show it is not one-to-one.
   - **Proof by counterexample:** Consider $f(5) = f(-5)$, but $5 \neq -5$, thus, f is not one to one.

Note: Recall that $f : A \to B$, where f is onto B if $\forall b \in B$, $|existsa \in A : f(a) = b$, so $im\ f = B$.

3. Show $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 4$ is onto.
   - **Proof:** Consider $b \in \mathbb{R}$. We need to find $a \in \mathbb{R}$ so that $f(a) = b$.
   - Let $a \in A : a = \frac{1}{3}(b - 4)$
     Then $f(a) = f(\frac{1}{3}(b - 4)) = 3(\frac{1}{3}(b - 4)) + 4 = b$
     Thus, f is onto.

4. **Proposition:** Let f be a function. The inverse relation f^{-1} is a function if and only if f is one-to-one.
   - **Proof:** Let f be function
   - $\Rightarrow$ Assume f^{-1} is a function. We need to show that f is one-to-one.
   - Suppose $f(x) = f(y)$, then $\exists z : (x, z), (y, z) \in f$. Then $(z, x), (z, y) \in f^{-1}$ by the definition of the inverse relation. And because it is a function, then $x = y$ by definition.
   - Therefore, f is one-to-one.
   - $\Leftarrow$ Assume f is one-to-one. We need to show f^{-1} is a function.
   - Let $(x, y), (x, z) \in f^{-1}$ and we need to show $y = z$.
   - Then $(y, x), (z, x) \in f$, so $f(y) = f(z) = x$ and f is one-to-one, so $f(y) = f(z) \Rightarrow y = z$. Thus, f^{-1} is a function.

2 Pigeonhole Principle:

Let A and B be finite sets and let $f : A \to B$

- If $|A| > |B|$, then f is not one-to-one.
- If $|A| < |B|$, then f is not onto.
- If $f : A \to B$ is one-to-one, then $|A| \leq |B|$
- If $f : A \to B$ is onto, then $|A| \geq |B|$
- If there is a bijection for $f : A \to B$, then $|A| = |B|$
- Let A and B be finite sets with $|A| = a$ and $|B| = b$.

  1. The number of functions from A to B is b^a
  2. If $a \leq b$, the number of functions $f : A \to B$ is $\frac{b!}{(b-a)!}$

Note: If $a > b$, then the number one-to-one functions is zero

  3. If $a \leq b$, the number of functions $f : A \to B$ is $\sum_{j=0}^{b}(-1)^j(b-j)^a$

Note: If $a < b$, the number of onto functions is zero.

  4. If $a = b$, the number of bijections for $f : A \to B$ is $a!$ or $b!$

Note: If $a \neq b$, the number of bijections is zero

2.1 Examples:

1. 24.1(b) $\{(x, y) : x, y \in \mathbb{Z}, y = 2x\}$ where $dom\ f = \mathbb{Z}$ and $im\ f =$ the set of all even integers.
   - Is f one-to-one? **Yes**
   - What is f^{-1}? f^{-1} is not onto $\mathbb{Z}$, because $f^{-1}(x) = \frac{1}{2}x$.
2. (g) $\{(x, y) : x, y \in \mathbb{Q}, x^2 + y^2 = 1\}$ is not a function, because $(0, 1) \in f$, but $(0, -1) \in f$.
3. (j) $\{(x, y) : x, y \in \mathbb{N}, \binom{x}{y} = 1\} = f$ is not a function, because $\binom{x}{0} = 1, \binom{x}{x} = 1$, so $(x, 0) \in f$, but $\binom{x}{0} \in f$.
4. 24.4 Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Write down all functions $f : A \to B$, indicate which are one-to-one, and which are onto B.
   - $\{(1, 3), (2, 4)\}$: onto ✓, one-to-one ✓
   - $\{(1, 4), (2, 3)$: onto ✓, one-to-one ✓
   - $\{(1, 4), (2, 4)$: onto ×, one-to-one ×
   - $\{(1, 4), (2, 3)$: onto ×, one-to-one ×

2.2 What is the Pigeonhole Principle?

Dirichlet Box Principle: States that if $n + 1$ or more pigeons are placed in n holes, then one hole must contain two or more pigeons.

2.3 Examples:

1. Assume you have an infinite number of red, green, blue, and black socks in a drawer. How many socks do you need to pull out of the drawer to guarantee a pair?
   A: You will need to pull out 5 socks to guarantee a pair. In this case, pigeons are socks, which you pull out, and the holes are the colors of the socks.
2. Let $n \in \mathbb{N}$. Then there exist positive integers a and b, with $a \neq b$, such that $n^a - n^b$ is divisible by 10.
   A:Any natural number is divisible by 10 if its last digit is zero.
   - **Proof:** Consider 11 natural numbers.
   $$n^1, n^2, ..., n^{11}$$
   - Since there are ten possible one's digits, but we have 11 different numbers, then two of these numbers must have the same last digits. Thus, $10|n^a - n^b$.

3 Composition

Let A, B, C be sets and let $f : A \to B$ and $g : B \to C$. Then the function $g \circ f$ is a function from A to C, defined by $(f \circ g)(a) = g(f(a))$, where $a \in A$, and gf is called the composition of g and f. Where $dom(g \circ f) = f$ and $g \circ f \neq f \circ g$

Let f and g be functions. To prove $f = g$

- Prove $dom\ f = dom\ g$
- Prove that for every x in their common domain, $f(x) = g(x)$

Identity Function:
Definition: Let A be a set. The identity function on A Is the function $id_A(a) = a$, so $id_A = \{(a, a) : a \in A\}$.

3.1 Examples:

1. Let $A = \{1, 2, 3, 4, 5, 6\}$ and let $f : A \to A$, and $g : A \to A$. Find $g \circ f$ and $f \circ g$

$$f = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$$

$$g = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$$

   Therefore,

$$g \circ f = \{(1, 5), (2, 5), (3, 5), (4, 5), (5, 5)\}$$
$$f \circ g = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$$

2. Let A $f : A \to B, g : B \to C, h : C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f$

Note: Composition of functions are associative. However, composition of functions are not commutative.

3. Let A and B be sets and $f : A \to B$. Then $f \circ id_A = id_B \circ f = f$

   - **Proof:** Consider $f \circ id_A$ and f. We know that $dom\ (f \circ id_A) = dom\ id_A = A = dom\ f$, so all domains are the same. Let $a \in A$, then $(f \circ id_A)(a) = f(id_A(a)) = f(a)$, so $f \circ id_A$ and f have the same image $\forall a \in A$. Thus, $f \circ id_A = f$.
   - Similar argument for $id_B \circ f$. $dom(id_B \circ f)(a) = id_B(f(a)) = f(a)$, so $id_B \circ f$ and f have the same integers. Thus $id_B \circ f = f$
   - Let $f : A \to B$ be a bijection, then $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$.