Lecture 10 - Discrete Mafematiks

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1 Induction

1.1 Examples:

1. Prove by induction that $2^n \leq 2^{n+1} - 2^{n-1} - 1$ where $n \in \mathbb{N}$

Proof: **Base Case:** Let $n = 1$, so we have $2^1 \leq 2^{1+1} - 2^{1-1} - 1$ ✓

- **Inductive Hypothesis:** Suppose the above inequality is true for $n = k$.

$$2^k \leq 2^{k+1} - 2^{k-1} - 1$$

We need to show that the given inequality is true for $n = k + 1$.

- Consider:

$$2^{k+1} \leq 2^{k+1} - 2^{k-1} - 1$$
$$2 \cdot 2^k \leq 2^{k+2} - 2^k - 1$$
$$2 \cdot 2^k \leq 2 \cdot (2^{k+1} - 2^{k-1} - 1) \leq 2^{k+2} - 2^k - 1$$
$$2 \cdot 2^k \leq 2 \cdot (2^{k+1} - 2^{k-1} - 1) = 2^{k+2} - 2^k - 1 \leq 2^{k+2} - 2^k - 1$$

Which is true for $n = k + 1$.

2. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{2^n} \geq 1 + \frac{n}{2}, n \in \mathbb{N}$

Proof: **Base Case:** Let $n = 1$, so $1 + \frac{1}{2^1} \geq 1 + \frac{1}{2}$, and $\frac{3}{2} \geq \frac{3}{1}$

- **Inductive Hypothesis:** Assume this inequality is true for $n = k$, then $1 + \frac{1}{2} + ... + \frac{1}{2^k} \geq 1 + \frac{1}{2}$ We need to prove it holds true for

$$1 + \frac{1}{2} + ... + \frac{1}{2^k} + \frac{1}{2^{k+1}} \geq 1 + \frac{1}{2} + \frac{1}{2^{k+1}}$$

$$1 + \frac{1}{2} + ... + \frac{1}{2^k} + \frac{1}{2^{k+1}} \geq 1 + \frac{1}{2} + \frac{1}{2^{k+1}} = 1 + \frac{k}{2} + \frac{1}{2^k \cdot 2} =$$

2 Recurrence Relations

Definition: When terms of a sequence depend on the previous terms.

First-Order Recurrence: $a_0 = S_{n-1}$, where a_0 in given and $S =$ some number.

Remark: The list of as sequence can be represented int he $1st$-order recurrence

$$a_n = S \cdot a_{n-1} + t$$

and in terms of another formula which contains the $S^n th$ term.
All solutions to the recurrence relation $a_n = S_{n-1} + t$ where $S \neq 1$ have the form

$$a_n = C_1 \cdot S^n + C_2$$

where C_1 and C_2 are specific numbers a_0 is given.

The solution to the recurrence relation of the form

$$a_n = a_{n-1} + t$$

where $S = 1$ is

$$a_n = a_0 + nt$$

Second-order recurrence: When a_n depends on 2 previous terms of a sequence, so

$$a_n = S_1 \cdot a_{n-1} + S_2 \cdot a_{n-1}$$

Where S_1 and S_2 are constants and a_{n-1} and a_{n-2} are given is $a_n = r^n$ to the second order recurrence

$$a_n = S_1 a_{n-1} + S_2 \cdot a_{n-2}$$

if there is only one solution.

However, if there are two distinct solutions, $r_1 \neq r_2$, then the solution is

$$a_n = C_1 \cdot r_1^n + C_2 \cdot r_2^n$$

where C_1 and C_2 we find solving a system of equations using a_0 and a_1

Let S_1 and S_2 be numbers, so that $x^2 - S_1 x - S_2 = 0$ has exactly one root $r \neq 0$, then every solution to the recurrence relation $a_n = S_1 \cdot a_{n-1} + S_2 \cdot a_{n-1}$ is of the form $a_n = C_1 \cdot r^n + C_2 \cdot n \cdot r^n$

Note: Sequences can be generated by polynomials

$$0^2 + 1^2 + ... + n^2 = \frac{(2n+1)(n+1) \cdot n}{6}$$

if a sequence is generated by a polynomial, we should be able to recognize that and if possible, find the polynomial.
The difference operator Δ is when we need to subtract one term of a sequence from another, so $\Delta a = a_n - a_{n-1}$, then we get a new sequence which will have less terms in it.

2.1 Examples:

1. Consider $a_n = 2 \cdot a_{n-1} + 3$ and $a_0 = 1$. Find the first 6 terms.

   $$5, 13, 29, 61, 125, 253$$

   Where the solution for this recurrence is $a_n = 4 \cdot 2^n - 3$

2. Solve the recurrence:

   $$a_n = 5 \cdot a_{n-1}$$
   $$1, 8, 43$$

   - The solution is in the form $a_n = C_1 \cdot S^n + C_2$ where we need to solve for C_1 and C_2 using $a_0 = 1$ and $a_2 = 8$.
   - Solve the below system of equations:

   $$a_0 = 1 = C_1 \cdot 5^0 + C_2$$
   $$a_1 = 8 = C_1 \cdot 5^1 + C_2$$
   $$C_1 = \frac{7}{4}, C_2 = \frac{-3}{4}$$
   $$a_n = \frac{7}{4} \cdot 5^n - \frac{3}{4}$$

3. 23.1(b) Find the solution for the recurrence relation

   $$a_n = a_{n-1} + 3$$
   $$soln = 5 + 3n$$

4. Solve the following recurrence
$$a_n = 3 \cdot a_{n-1} + 4 \cdot a_{n-2}$$
when $a_0 = 3, a_1 = 2$.
   - We need to solve
$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1) = 0$$
   - The roots are $r = 4$ and $r = -1$
$$a_n = C_1 \cdot 4^n + C_2 \cdot (-1)^n$$
   - Now, we need to solve for C_1 and C_2
$$a_0 = 3 = C_1 \cdot 4^0 + C_2(-1)^0$$
$$a_1 = 2 = C_1 \cdot 4^1 + C_2(-1)^1$$
$$C_1 = 1, C_2 = 2$$
   - So $a_n = 4^n + 2 \cdot (-1)^n$ is the solution
5. $a_n = n^3 - 5n + 1$, then
$$\Delta a_n = a_{n+1} - a_n$$
$$= [(n+1)^3 - 5(n+1) + 1] - [n^3 - 5n + 1]$$
$$= n^3 + 3n^2 + 3n + 1 - n^3 + 5n - 1$$
$$= 3n^2 + 3n - 4$$
   - If a sequence $\{a_n\}$ is generated by a polynomial of degree d, then Δ^{d+1} in the all zero sequence

3 Functions

Defintion: A relation f is called a function if $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$
Remark: So it is a *one-to-one* function, so every x must be "mapped" only to one y.

- There must be *bijection* which is "onto" and "one-to-one".
- Functions also can be defined as sets of ordered pairs.
- The set of all possible first elements of the ordered pairs $(a, b) \in f$ is called the domain of f and denoted "*dom f*"
- The set of all possible second elements of the ordered pairs $(a, b) \in f$ is called the image of f and denoted "*im f*"
$$f : A \to B_{x \mapsto y}$$
- for sets $A = dom\ f$ and $B = im\ f$, x gets mapped to y. We say f is a function from A to B, provided $A = dom\ f$ and $im\ f \subseteq B$, so we could say f is a mapping or map from A to B.

Note: "one-to-one" is injective.

To show: $A \to B$

- Prove that f is a function
- Prove that $dom\ f = A$
- Prove that $im\ f \subseteq B$

Let A and B be a finite set with $|A| = a$ and $|B| = b$, then the number of functions from A to B is b^a.
A function is called *one-to-one*, provided that whenever $(x, b), (y, b) \in f$, then $x = y$. Or, if $x \neq y$, then $f(x) \neq f(y)$. If f is one-to-one, then an inverse function, f^{-1}, exists.
Let $f : A \to B$, we say f is onto (or surjective) onto B, provided that for every $b \in B$, there is an $a \in A$, so that $f(a) = b$, so $im\ f = B$. A bijection is when a map or function is both onto and one-to-one.

3.1 Examples:

1. $G = \{(5,3),(5,-11),(9,4),(2,1)\}$

$$5 \mapsto 3, 4 \mapsto -11, 9 \mapsto 4, 2 \mapsto 1$$

   - This is not ”one-to-one”
   - 3 is an image of 5, OR 5 is a pre-image of 3