Lecture 1 - Discrete Mafematiks

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1 Numbers

$\mathbb{N} \rightarrow$ We have natural numbers, {1,2,3,4,...}. also the same as: $\mathbb{Z}^+$.

1. We have whole numbers {0,1,2,3,4,...}

$\mathbb{Z} \rightarrow$ We have integers {-∞,...,-2,-1,-,1,2,3,...,∞}

$\mathbb{Q} \rightarrow$ We have rational numbers $\frac{n}{m}$ where $m \neq 0$.

$\mathbb{Q}^c \rightarrow$ complement are irrational numbers

$\mathbb{C} \rightarrow$ the set of all complex numbers, where if $\mathbb{Z}$ is a complex number of the following form $z = a + bi$ where a and b are any real numbers and $i = \sqrt{-1}$. **Note:** All real numbers are a subset of complex numbers.

   (a) All real numbers are complex numbers. A is called a real part of z and notation is $Re(z) = a$.
   (b) *Example:* $z = 3 + 2i$.
   (c) $Re(z) = 3$ and $Im(z) = 2$.

Note: A conjugate of complex number $z = a + bi$ is $\bar{z} = a - bi$

2. if $z = \bar{z}$ then it is a real number.
3. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$

2 Division

1. We say that a number divides another in the form of 5|25. This means that 5 *divides* 25. It's equivalent is 25/5.
   (a) If we say that $x|25$, we say that x is a factor of 25.
   (b) If we say that $5|x$, we say that 5 is a divisor of x.
2. If $y = 2a$ then y is even, because $2a|y$.

Note: An integer p is called prime if $p > 1$ and the only positive divisions of p are 1 and p.

Note: A composite integer is an integer for which there exists an integer b, so that $1 < b < a$ and $b|a$

3 Theorems

a) A theorem is a declarative statement which needs to be proved.

b) An axiom is a statement that does not require proof and can be used as fact.

c) Theorems can vary in purpose and size.

d) Claims, Propositions, and Lemmas are *small* theorems which are used for proving.

e) A *Corollary* is a theorem which is derived as a conclusion of a large theorem.

Note: Every detail matters, we have to understand where everything comes from.

Proofs: are logically derived statements which prove theorems. We must use definition and axioms in proofs.

4 Proof Structure

1. If A, then B = $A \to B$ (see Table 1)
2. If and only if = $A \Leftrightarrow B$ (see Table 2)
3. A and B= $A \cap B$ (see Table 3)
4. A and B= $A \cup B$ (see Table 4)

Table 1: If A, then B

A	B	$A \Rightarrow B$
True	True	True
True	False	True
False	True	False
False	False	True

Table 2: A iff B

A	B	$A \Leftrightarrow B$
True	True	True
True	False	False
False	True	False
False	False	True

Table 3: A ∩ B

A	B	$A \cap B$
True	True	True
True	False	False
False	True	False
False	False	False

Table 4: A ∪ B

A	B	$A \cup B$
True	True	True
True	False	True
False	True	True
False	False	False

4.1 Vacuous Truths

1. Vacuous Truth $\to$ "If A, then B," which is a statement that is impossible to check.
2. This statement is considered to be true.
3. As per Wikipedia, "a vacuous truth is a conditional or universal statement that is only true because the antecedent cannot be satisfied."

4.2

Proposition: The sum of two even integers is even.

Proof: Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. Since x and y are even integers, then $2|x$ and $2|y$, so there exists arbitrary positive integers a and b, so that $x = 2a$ and $y = 2b$.

Then, $x + y = 2a + 2b = 2(a + b) = 2c$ where $c = a + b$ is some positive integer. $2(a + b)$ suggest that there exists some integer $c = a + b$ such that $x + y = 2c$. $2|(x + y)\square$

Proposition: (5.1) The sum of two odd integers is even.

Proof: Let x and y be odd integers. $x = 2a + 1$ and $y = 2a + 1$ where a and b are some integers.

Then, $x + y = (2a + 1) + (2b + 1) = 2(a + b) + 2 = 2(a + b + a)$. Let $c = a + b + 1$, where c is an integer. Because a and b are integers, then $x + y = 2(a + b + 1) = 2c$, so $2|(x + y)$.

Proposition: (5.6) Prove that the product of two odd integers is odd.

Proof: Let x and y be odd integers. $x = 2a + 1$ and $y = 2a + 1$ where a and b are some integers.

Then, $x \cdot y = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(ab + b + a) + 1$. Let $c = 2ba + a + b$, where c is an integer. Because a and b are integers, then $x = 2(2ab + a + b) + 1 = 2c + 1$, so $2 + 1|(x)$.

Proposition: Let a, b, c, d be integers. If $a|b$, $b|c$ and $c|d$, then $a|d$.

Proof: Since $a|b$, then there exists an integer x, such that $b = ax$.

Similarly, there exists an integer y such that $c = by$, and an integer z such that $d = cz$.

$c = by = ax(y)$ and since $d = (ax(y))z$, let $w = xyz$ which is an integer, so $d = ((ax(y))z = a(xyz) = a \cdot w$. Therefore, $a|d$ $\square$.

Proposition: Let x be an integer. If $x > 1$, then $x^3 + 1$ is composite.

Recall: $x^3 + 1 = (x + 1)(x^2 - x + 1) and a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Proof(1): Since $x^3 + 1 = (x + 1)(x^2 - x + 1)$, and $x^2 - x + 1 = c$ where c is an integer because x is an integer, then $x^3 + 1 = (x + 1)(x^2 - x + 1) = (x + 1) \cdot c$, so $(x + 1)|x^3 + 1$

Note: We need to show that $1 < x + 1 < x^3 + 1$, because based on the definition of a composite number, $1 < b < a$.

Proof(2): Since $x > 1$, $x < x^3$, and $x + 1 < x^3 + 1$, so $1 < x + 1 < x^3 + 1$.

Therefore, $x + 1$ is the divisor of $x^3 + 1$ (1) and $1 < x + 1 < x^3 + 1$ (2) is composite, then the statement is True. $\square$

Proposition: Let x be an integer. Then x is even, if and only if x is odd.

Remark: Proofs of "if and only if" must consist of two parts. (1) $A \Rightarrow B$ and (2)$B \Rightarrow A$.

Proof(1): If x is even, then $x + 1$ is odd. Let x be an even integer. Then there exists some integer a such that $x = 2a$.

Adding "1" to both sides leads to $x + 1 = 2a + 1$ which is odd.

Proof(2): Let $x + 1$ be an odd integer. Then there exists an integer b such that $x + 1 = 2b + 1$.

Subtracting "1" from both sides leads to x. $x = 2b$, therefore x is divisible by 2. x is even.

$\square$