Assignment 3

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(1) Prove by induction

(a) (8 points) $1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

- Let $n \in N$. Let $n = 0$, so we have $0^3 = 0 = \frac{0^2(0+1)^2}{4}$ ✓
- Let $n = 1$, so we have $1^3 = 1 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$ ✓
- We can assume that the equation will hold for $n = k$

$$1^3 + 2^3 + ... + k^3 = \frac{k^2(k+1)^2}{4}$$

Now we have to show that the given equation is true for $n = k + 1$

- Consider:

$$1^3 + 2^3 + ... + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$
$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k^2+2k+1)(k^2+4k+4)}{4}$$
$$\frac{k^2(k^2+2k+1)}{4} + (k^3+3k^2+3k+1) = \frac{(k^4+6k^3+13k+4)}{4}$$
$$\frac{k^4+2k^3+k^2}{4} + \frac{(4)(k^3+3k^2+3k+1)}{4} = \frac{(k^4+6k^3+13k+4)}{4}$$
$$\frac{(k^4+6k^3+13k+4)}{4} + = \frac{(k^4+6k^3+13k+4)}{4} ✓$$

- Therefore, by the mathematical process of induction, the proposition holds □

(b) (8 points) $2^n \leq 2^{n+1} - 2^{n-1} - 1$

- Let $n \in N$. Let $n = 1$, so we have $2^1 = 2 \leq 2^{1+1} - 2^{1-1} - 1 = 2$ ✓
- We can assume that the equation will hold for $n = k$.

$$2^k \leq 2^{k+1} - 2^{k-1} - 1$$

Now we have to show that the given inequality is true for $n = k + 1$

- Consider:

$$2^{k+1} \leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1$$
$$2^{k+1} \leq 2^{k+2} - 2^k - 1$$
$$(2) \cdot 2^k \leq 2 \cdot (2^{k+1} - 2^{k-1}) - 1$$
$$(2) \cdot 2^{k+1} - 2^{k-1} - 1 \leq 2 \cdot (2^{k+1} - 2^{k-1}) - 1 ✓$$

- Therefore, by the mathematical process of induction, the proposition holds □

(2) (6 points) Solve the following recurrence relations by stating an explicit formula for a_n, and use your result to calculate a_9.

(a) $a_n = 3a_{n-1} - 1,\ a_0 = 10$

$$29, 86, 257, ...$$

$$a_n = 9.5 \cdot 3^n + \frac{1}{2}$$

$$a_0 = 10 = C_1 \cdot 3^0 + C_2$$

$$a_1 = 29 = C_1 \cdot 3^1 + C_2$$

$$a_9 = 9.5 \cdot 2^9 + \frac{1}{2} = 186,989$$

(b) $a_n = 8a_{n-1} - 15a_{n-2},\ a_0 = 1,\ a_1 = 4$

$$x^2 - 8x + 15 \Rightarrow x = 3, x = 5$$

$$a_n = C_1 \cdot 3^n + C_2 \cdot 5^n$$

$$a_0 = 1 = C_1 \cdot 3^0 + C_2 \cdot 5^0$$

$$a_1 4 = 3 \cdot 3^1 + C_2 \cdot 5^1$$

$$C_1 = \frac{1}{2}, C_2 = \frac{1}{2}$$

$a_n = \frac{1}{2} \cdot 3^n + \frac{1}{2} \cdot 5^n$
$a_9 = \frac{1}{2} \cdot 3^9 + \frac{1}{2} \cdot 5^9 = 986,404$

(3) (12 points) Let A and B be finite sets and let $f : A \to B$. Prove that any two of the following statements being true implies the third.

(i) f is one-to-one, (ii) f is onto, (iii) $|A| = |B|$

A Let f be defined as the function that is (i) and (ii). For $a \in A, b \in A$, beacuse f is (i), their images cannot be the same, so $f(a) \neq f(b)$. We know that there will two unique images of B for every element two elements in A. Therefore, we reach $|A| \leq |B|$.

- If we let $b \in B$, because the function is (ii), we know that there will exists at least one element $a \in A : f(a) = b$. Therefore, there is at least one element in A that matches up with every element in B. Therefore, we reach $|A| \geq |B|$.

$$|A| \geq |B| \wedge |A| \leq |B| = |A| = |B|$$

Therefore, the two sets contain the same number of elements.

B Let f be the function that is (ii) and (iii).

- We know because f is onto that for every $b \in B$, there will be a corresponding element in A. Because they contain the same number of elements, and we know that two elements cannot have the same image, f is one-to-one.

C Let f be defined as the function that is (i) and (iii).

- We know because f is one-to-one, that there are no two element sin A that have the same image. Because A and B contain the same number of elements and no two elements can have the same image, the function must be onto.

$\Rightarrow$ Because **A,B,C** hold true, this statement is true. $\square$

1. (6 points) Prove by the smallest counterexample that $n < 2^n$ for all $n \in N$.

   Proof: Consider some $n \in N$, then $n < 2^n$

   - Suppose for the sake of contradiction that this statement is false. Then the set of all possible counterexamples is defined by the following $X = \{k \in N : k \not< 2^k\}$
   - Note that $k \neq 0$, since $0 < 2^0$, so $k > 0$. Then for $k-1$, the statement is true.
   - Thus, $(k-1) = 2^{k-1}$
   $$(k-1) \not< 2^{k-1}$$
   - Add 1 to both sides, and remember that $x-1 > 0$, $2^{x-1} > 1$
   $$k - 1 + 1 \not< 2^{k-1} + 1 = 2^{k-1} + 2^{k-1} = 2 \cdot 2^{k-1} = 2^{k-1+1} = 2^k \Rightarrow\Leftarrow$$

   However, the last statement implies that k satisfies the statement, which is a contradiction with the counterexample that we proposed.

   Therefore, because our supposition that the statement is false is false, our statement is true. $\square$

2. (10 points) Let A and B be sets. Prove that $A = B$ if and only if $id_A = id_B$

   $\Rightarrow$ **Proof:** If $A = B$, then $A \subseteq B \wedge B \subseteq A$. Let $a \in A$. We know that $\forall a \in A, a \in B$.

   $\Leftarrow$ If $A \subseteq B \wedge B \subseteq A$, then $id_A(b) = \{(b, a) : b \in A\} = a$, and $id_B(a) = \{(b, a) : a \in B\} = b$.

   - If $A \subseteq B \wedge B \subseteq A$, $dom\ (id_B)(a) = dom\ (id_A)(b) = id_B(f(a)) = f(a)$, so A and B have the same integers. $\square$