Assignment 2

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(1) (3 points) Prove that the equality relation on the set of integers is antisymmetric.

- $\forall x, y \in Z, (x = y) \wedge (y = x) \Rightarrow x = y$, therefore the equality relation on the set of integers is antisymmetric $\square$.

(2) (12 points) Let us say that two integers are *near* one another provided the absolute value of their difference is 2 or smaller. For example, 3 is near to 5, and 10 is near to 9, but 8 is not near to 4. Let R stand for this *is-near-to* relation. Please do the following:

- write down R as a set of ordered pairs:

$$R = \{(x - y) : |(x - y)| <= 2\}$$

- Prove or disprove: R is reflexive.
  - R is reflexive because $\forall x \in R, xRx$ and $(x - x) = 0 <= 2$.
- Prove or disprove: R is irreflexive.
  - R is **not** irreflexive because $\forall x \in R, xRx$, and $(x, x) \in R$.
    Counterexample: $1R1$ is a possibility.
- Prove or disprove: R is symmetric.
  - R is symmetric because $\forall x, y \in R, xRy \Rightarrow yRx$, also $|x - y| = |y - x|$
- Prove or disprove: R is antisymmetric.
  - R is **not** antisymmetric because $\forall x, y \in R, (xRy \wedge yRx) \Rightarrow x \neq y$.
    Counterexample: $|2 - 4| \leq 2 \wedge |4 - 2| \leq 2$, but $2 \neq 4$.
- Prove or disprove: R is transitive.
  - R is **not** transitive because $\forall x, y \in R, (xRy \wedge yRz) \Rightarrow x \not{R} z$.
    Counterexample: $|2 - 4| \leq 2 \wedge |4 - 6| \leq 2$ but $|2 - 6| \not\leq 2$.

(3) (8 points) Prove: A relation R on a set A is antisymmtric if and only if

$$R \cap R^{-1} \subseteq \{(a, a) : a \in A\}$$

"$\Rightarrow$" If $R \cap R^{-1}$ is $\emptyset$, then $R \cap R^{-1} \subseteq \{(a, a) : a \in A\}$ because emptyset is a subset of all sets.

- If $R \cap R^{-1}$ is not an emptyset, then $(a, b) \in R \cap R^{-1}$. This means that $(a, b) \in R$ and $(a, b) \in R^{-1}$. If we use the definition of the inverse relation, $(a, b) \in R$, and $(b, a) \in R^{-1}$. If $(a, b) \in R$, and $(b, a) \in R^{-1}$ then $a = b$. We see that $R \cap R^{-1} \subseteq \{(a, a) : a \in A\}$.

"$\Leftarrow$" Assume that $R \cap R^{-1} \subseteq \{(a, a) : a \in A\}$.

- We know that $(a, b) \in R$ and $(b, a) \in R^{-1}$. If an element is in $R \cap R^{-1}$, we know that $(a, b) \in R \cap R^{-1}$ and $(b, a) \in R \cap R^{-1}$. Since $R \cap R^{-1} \subseteq \{(a, a) : a \in A\}$, then $(a, b) \subseteq \{(a, a) : a \in A\}$ and $(b, a) \subseteq \{(a, a) : a \in A\}$. Therefore $a = b$, and R is antisymmetric. $R \cap R^{-1} \subseteq \{(a, a) : a \in A\}\square$.

(4) (8 points) Let R be an equivalence relation on set A. Prove that the union of all of R's equivalence classes is A. By the definition of an equivalence relation, we know that the union of all of the equivalence classes is an equivalence relation because the equivalence classes are mutually disjoint.

- $\exists a \in A : x \in [a]$ where $[a]$ is the set of all elements that are related to a. So $x \in A$. $x \in \cup_{a \in A}[a] \subseteq A$.
- Let $x \in A$. If R is an equivalence relation, then by reflexivity, we know that $\forall x \in A, xRx \in R$. x must be an element of A.
- (i) $x \in \cup_{a \in A}[a] \subseteq A$ by the definition of a subset.
- Because R is an equivalence relation, R is reflexive, and $xRx \in R$. This implies that x in in the equivalence class $[x]$.
- (ii) Since $\cup_{a \in A}[a]$ is the union of all equivalence classes, the union must contain $[x]$ and therefore contains x.
- $x \in A \Rightarrow x \in \cup_{a \in A}[a], A \subseteq \cup_{a \in A}[a]$ by the definition of a subset.
- because (i) and (ii), we see that $\cup_{a \in A}[a] = A\Box$

(5) (a) (3 points) How many partitions, with exactly two part can be made of the set $1, 2, 3, 4$? and for $1, 2, 3, .., 100$?

A: There will be 2^n total subsets of n elements. If we want to get the amount of partitions of n into two subsets, we will use the equation 2^{n-1}, accounting for the emptyset. We will also get one leftover subset that would contain the entire original set, so our final equation is $2^{n-1} - 1$.

$\Rightarrow \{1, 2, 3, 4\} : 2^{4-1} - 1 = 7$

$\Rightarrow \{x : 0 \leq x \leq 100\} = 2^{100-1} - 1 = 6.338253001e29$

(b) (3 points) In how many different ways can we partition an n-element set into two parts if one part has four elements and the other part has all the remaining elements?

A: $\binom{n}{4}$ because we are selecting a 4 element subset from n choices. Our 4-element subset will partition our n element sets.

(c) (3 points) A poker hand consists of 5 cards chosen from a standard deck of 52 cards. How many different poker hands are possible?

A: $\binom{5}{52}$

(6) (7 points) Let n be a positive integer. Give a combinatorial proof that

$$n^2 = n(n-1) + n$$

$\Rightarrow$ **Combinatorial Proof:** Consider the set $A = \{1, 2, 3, ..., n\}$

- **Question:** How many ways can we pick a list of 2 elements from A?